

I. Ancient Times

All the major ancient civilizations developed around river valleys. By 2000 BC, there were civilizations thriving around the Nile (Egypt), the Tigris and Euphrates (Babylon), the Ganges (India) and the Yangtze (China). In all of the societies, the periodic flooding of the region made the ground very fertile and enabled successful agricultural activities. But in order to know when the river might flood, how to redistribute the land after all markings had been washed away, and how to manage the irrigation systems required extensive mathematical know-how. Essentially, this is how mathematics was born.

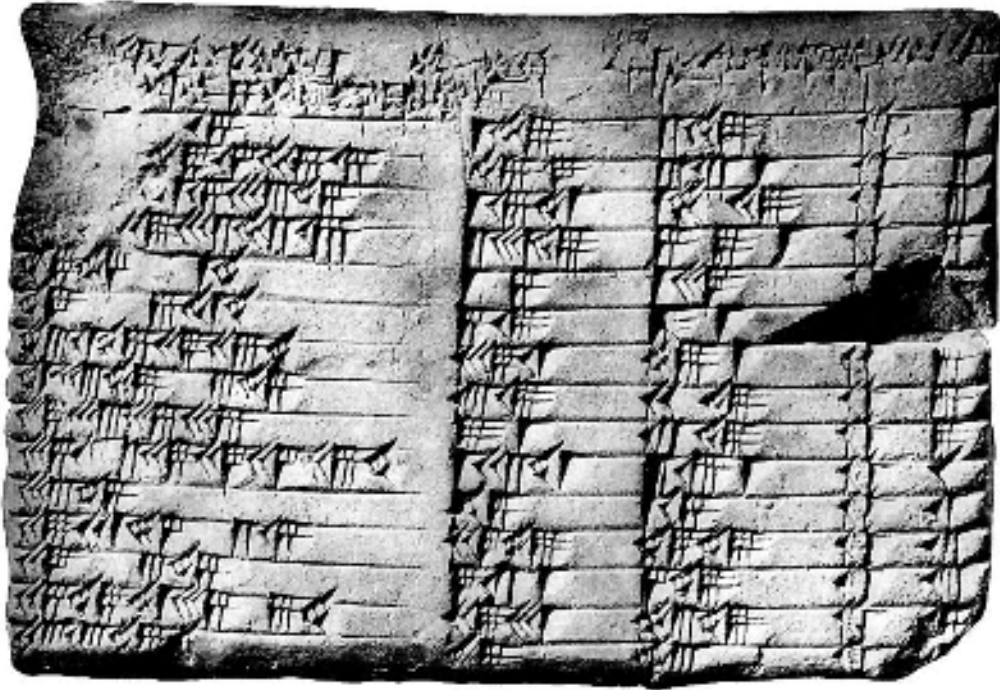
While all these early civilizations did mathematics, it was the Babylonians and Egyptians that had the greatest success recording and preserving their accomplishments. The Babylonians used baked clay tablets to record their knowledge and these have proven to be virtually imperishable (although the sometimes breakable). The Egyptians wrote their achievements on papyri and these have also survived quite well due to the extremely dry climate of their region. Conversely, the Indians and Chinese wrote on bark or bamboo and their wet climate have undoubtedly destroyed many records. China additionally had another obstacle that prohibited the survival of ancient documents. China is an extremely old civilization, but periodic turnover in dynasties led to the destruction of all previous knowledge. The new emperor would routinely want all knowledge to come from him or be attributed to his rule. Therefore, very few texts or documents have survived.

Section 1: Egypt and Babylon

I named this section “Egypt and Babylon” because the surviving documents from Egypt are older. But I’m going to discuss Babylon first...so sue me.

Babylon was a civilization that thrived in modern day Iraq, in the Fertile Crescent between the Tigris and Euphrates rivers. They were very advance mathematically and also quite unique in that they used a sexagesimal numeration system (base 60). It is to the Babylonians that we owe the 360 degrees of a circle for instance. But they were also quite accomplished at algebra, geometry, and number theory.

The Babylonians recorded their knowledge and achievements on clay tablets in cuneiform and then baked them in the sun to dry and preserve them and this method has proven very successful. Over 500,000 such tablets have been uncovered so far with maybe millions more still buried. Of the half a million discovered so far, around 400 are explicitly mathematical. Many contain tables of information such as multiplication tables, and tables of reciprocals, squares, and cubes. Most tablets come from the time of around 2100 BC to 1600 BC (although many more recent tablets from 600 BC or so also exist). These tablets show that the sexagesimal system was already well established. One in particular, *Plimpton 322*, is quite interesting. It is named for its catalogue number in the Plimpton Collection at Columbia University. Publisher George Plimpton purchased the tablet around 1922 and left his entire collection to Columbia University in the 1930’s. On the following page is an image of the tablet:



This tablet has four columns of data, but let's focus on the first three (from right to left). All the cuneiform symbols are again in sexagesimal notation, but here is their translation:

119	169	1
3367	11521*	2
4601	6649	3
12709	18541	4
65	97	5
319	481	6
2291	3541	7
799	1249	8
541*	769	9
4961	8161	10
45	75	11
1679	2929	12
25921*	289	13
1771	3229	14
56	53*	15

A few notes:

- (1) There are a couple of physical breaks in the tablet at the top left and the middle of the right hand side.
- (2) The asterisks denote probable errors in the data. I'll explain more about this shortly.
- (3) The first column (far right) is clearly just a numbering of the rows.

(4) The other two columns appear to be rather random, but with much study, math historians have discovered that they are (with the four exceptional errors) the hypotenuse and a leg of integral-sided right triangles. In other words, in each row (reading from right to left) we have (a) the row number, (b) the length of a hypotenuse, and (c) the length of one of the sides. The other side could be easily computed, but only if the Pythagorean Theorem was known. So *Plimpton 322* very simply is a table of Pythagorean triples. For example, on row 1, the third side would have been 120 ($169^2 - 119^2 = 120^2$). The fact that the remaining side was not included seems to indicate that the author assumed the reader would be able to compute the third side on their own, so apparently the Babylonians knew the Pythagorean Theorem approximately 1000 years before the Pythagoreans “discovered” it.

Regarding the “errors”, there seem to be two different kinds. The errors on rows 9, 13, and 15 appear to be typos. For instance, on row 9, the sexagesimal notation for 541 would be “9,1” (9 60’s and 1). The correct value is likely 481 which is just “8,1”. This error could be a very simple slip of the cuneiform stylus. Similarly, on row 13, the value listed (25921) is the square of the correct value (161) and on row 15, the value listed (53) is half the correct value (106). The error on row two appears to be something else entirely however, and it has prompted much study. But the lesson here is that people make mistakes, even 4,000 years ago.

Finally, we should say a word about the mysterious fourth column (the far left column). This appears to be the secant of one of the angles in the right triangle, showing that the Babylonians were even familiar with rudimentary trigonometry.

The Babylonians were also skilled at algebra and geometry. Systems of equations make their first appearance on Babylonian stone tablets dating from about 300 BC. Here is an example (written in modern notation):

Example 1: Solve the system of equations: $x + y = \frac{13}{2}$ and $xy = \frac{15}{2}$.

Their method of solving such a system began with supposing that both x and y were “about” half of $\frac{13}{2}$. Namely, let $x = \frac{13}{4} + z$ and $y = \frac{13}{4} - z$ for some unknown value z . (Notice that these values add up to $\frac{13}{2}$). If we substitute these values into the other equation, we get

$$\begin{aligned} \left(\frac{13}{4} + z\right)\left(\frac{13}{4} - z\right) &= \frac{15}{2} \\ \frac{169}{16} - z^2 &= \frac{15}{2} \\ \frac{169}{16} - \frac{15}{2} &= z^2 \\ \frac{49}{16} &= z^2 \\ \frac{7}{4} &= z. \end{aligned}$$

Using this value for z , we find the solution to the original system $x = 5$ and $y = \frac{3}{2}$.

Notice that systems of this type came from geometrical questions. The related application involves finding a rectangle with a given semiperimeter ($x + y = a$) and area ($xy = b$).

Systems of equations, Pythagorean triples, and trigonometry are all very advanced topics for 3,000-4,000 years ago. However, one of the perhaps most basic elements of modern mathematics that is missing is that of proof. In the early days of mathematics, texts (i.e. papyri or clay tablets) rarely explained why a particular method for solving a problem worked. In fact, the method usually was not even described in generality. A particular problem was described and its solution was given. This is especially evident when we study Egyptian papyri.

The oldest mathematical documents in existence are two papyri from Egypt dating between 1850 BC and 1650 BC. The Moscow Papyrus and the Rhind (or Ahmes) Papyrus both give explicit details of the Egyptian mathematics of their time. The Moscow Papyrus, the older or the two dating from approximately 1850 BC, contains 25 problems of a very practical nature dealing primarily with arithmetic and geometry. The Rhind Papyrus appears to be a copy of an earlier papyrus, transcribed by a man named Ahmes, and contains 85 problems dating from around 1650 BC. In both of these documents, each problem spelled out a correct procedure using specific values. As mentioned above, there is no proof that the procedure works, no demonstration of why it works, and no explanation of why it is the one chosen. If someone had a similar problem to solve, the author apparently was instructing the reader to copy the technique. There does not appear to be any desire for generalizations or explanations for several hundred years.

Most of the problems on the Egyptian papyri are numerical in nature and quite simple. But they do reveal a couple of fascinating aspects of Egyptian mathematics. First, Egyptians performed multiplication and division by repeated doubling. Every number can be written as the sum of powers of 2 (a fact the Egyptians apparently knew quite well), and therefore to multiply a number by another number, they simply added up the appropriate multiples of the given number.

Example 2: Multiply 51 by 28.

Let's list the multiples of 51 first:

n	$n \times 51$
1	51
2	102
4	204
8	408
16	816
32	1632
64	3264

(Note that this repeated doubling is fairly simple even for 4,000 years ago.) Now, since $28 = 16 + 8 + 4$, we just need to add up the 4th, 8th, and 16th multiples of 51:

$$\begin{aligned}
51 \times 28 &= 51 \times (16 + 8 + 4) = (51 \times 16) + (51 \times 8) + (51 \times 4) \\
&= 816 + 408 + 204 \\
&= 1428
\end{aligned}$$

Another interesting feature of Egyptian mathematics is their use of unit fractions (fractions with a numerator of 1). They wrote all fractions as the sum of unit fractions except $\frac{2}{3}$. A given fraction was broken down similar to the above example so that it was the sum of fractions with numerators all powers of 2, and then tables were consulted listing the unit fraction decomposition for fractions of the form $\frac{2}{n}$. In fact, the problems on the Rhind Papyrus are preceded by such a table for all odd n from 5 to 101.

Section 2: China and India

Two other early civilizations thrived in China and India. The Chinese culture flourished starting around 1500 BC and the Indians around 300 BC. As mentioned above, in both of these ancient societies there were almost certainly earlier achievements, however, the climates of the regions in which they existed were very humid and wet. Their methods of recording history have proven to lack the longevity of the Egyptians and Babylonians. Also, political turmoil in both cultures occasionally caused the destruction of all previous documents.

Chinese mathematics is a very rich field of study. We could take several sections (and weeks) and devote it strictly to their achievements. But I won't. I will however mention a couple of interesting achievements/anecdotes. The Chinese used a decimal numeration system and they were among the first civilizations to use a symbol for zero. However, they didn't view it as a number right away, it was just a placeholder. Their number system was also remarkable because it was the first to incorporate negative numbers. One of the most noteworthy texts from China was *Nine Chapters on the Mathematical Art*. This important book was the dominant mathematical text in China for approximately 1500 years. Its origin is unknown but was probably written around 50 BC. It obviously consists of nice chapters dealing with a wide range of mathematical topics. Current research is beginning to understand that while Chinese mathematicians did not have a concept of proof like the Greeks did (and like we are accustomed in modern times), *Nine Chapters* does possess some explanations that indicate the ancient Chinese had an appreciation for demonstrations.

Another contribution of the Chinese is their remarkably good approximation of pi. As you know, pi is defined as the ratio between the circumference and the diameter of a circle. It is an irrational number and for many, many years mathematicians (and others) tried to find better and better approximations for it. In fact, even the Bible weighs in:

And he made a molten sea, ten cubits from the one brim to the other: it was round all about, and his height was five cubits: and a line of thirty cubits did compass it about. (I Kings 7, 23)

This is taking place around the building of Solomon's Temple in 950 BC and would indicate that pi was 3, a very poor estimate even in its day. As far back as the Babylonian and

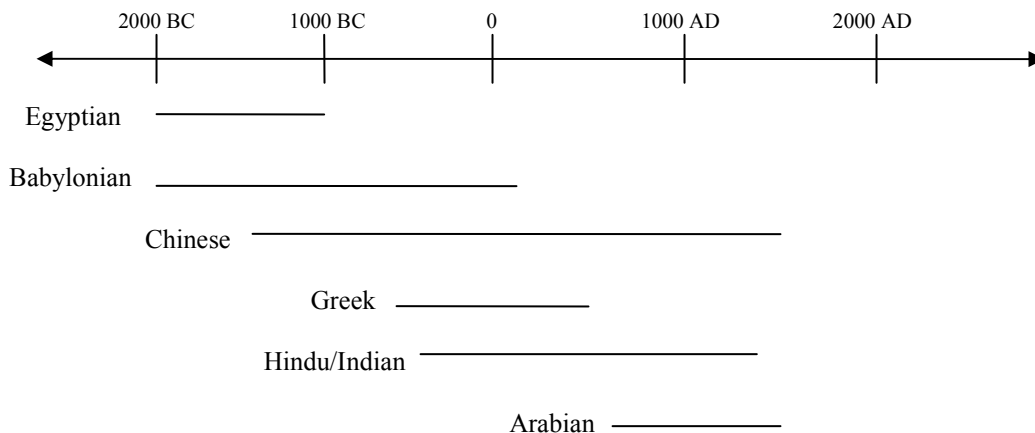
Egyptians, values such as 3.125 and $\sqrt{10}$ were being used. By 450 AD, the Chinese had $\pi \approx \frac{355}{113} = 3.141592$ which is accurate to 6 decimal places. This would not be improved for 1000 years. We will discuss this topic more when we talk about Archimedes and again during the European Renaissance.

Finally we owe magic squares to the Chinese. A magic square can be any size, but is usually 3×3 . The object is to write the nine integers 1 through 9 in a 3×3 grid so that every row, every column and both diagonals have the same sum. These have been very popular puzzles throughout history. Legend has it that around 2200 BC while the Emperor was standing by a lake, a turtle crawled ashore and upon his shell was a magic square. Cool.

The history of Hindu/Indian mathematics is similar to Chinese mathematics in that records are hard to come by and legends are hard to believe. One of their most important advances though was their use of the number 0. They were the first to consider it a number and to do arithmetic with it such as addition, subtraction, and multiplication. Division still stumped them. They believed for example that $\frac{a}{0} = \infty$ and $\frac{a}{0} \cdot 0 = a$. But perhaps their most important contribution to mathematics was their numerals which essentially became our numbers of today.

Section 3: Arabia

By studying Arabian/Persian mathematics at this point, we are definitely leaving the chronological order of things. If we put the various “ancient” cultures on a timeline we would get something like this:



We will cover the contributions of the Persians periodically over the next sections as they are relevant, but this seems to be the most appropriate place to cover the general influence of this culture. In 622 AD Mohammed made his flight from Mecca to Medina, and what followed was the unification of many Persian and Arabic tribes scattered around the Arabian peninsula and throughout the Middle East. Within a century, Islam was a dominant religion from the former Babylon (in the east) across northern Africa and even into Spain (in the west). There were many notable Persian mathematicians, and we will mention some of the most important later along

with their achievements, but the most important contribution of the Arabic world to mathematics was their preservation of ancient knowledge.

The time from approximately 500 AD to 1500 AD is very roughly called the Dark Ages. Obviously periods of human history do not start and stop quite as abruptly as we characterize when we look back. But in broad terms, between the end of the Greek civilization in 500 and the European Renaissance in 1500, intellectual advancements, innovation, and creativity came to a screeching halt. This too isn't exactly accurate. There were notable people in every century, but the Dark Ages were virtually devoid of achievements upon which we now rely...except for the Persians. As Moslem rulers would capture a town or a country, their scholars would set out translating all works of literature and knowledge into Arabic. Many Greek and Hindu achievements in particular would be lost for all time if we did not have their Arabic translations. It bears repeating, many Arabic mathematicians provided important contributions, but by preserving previous knowledge for future mankind, the culture did an amazing service.